On the geometry of the Humbert surface of square discriminant

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Isogenies of elliptic curves over $\mathbb{Q}$

Question: Which integers $m \geq 2$ can be the degree of a (cyclic) isogeny $\phi: E \to E'$ where E, E', and ϕ are defined over $\mathbb{Q}$?

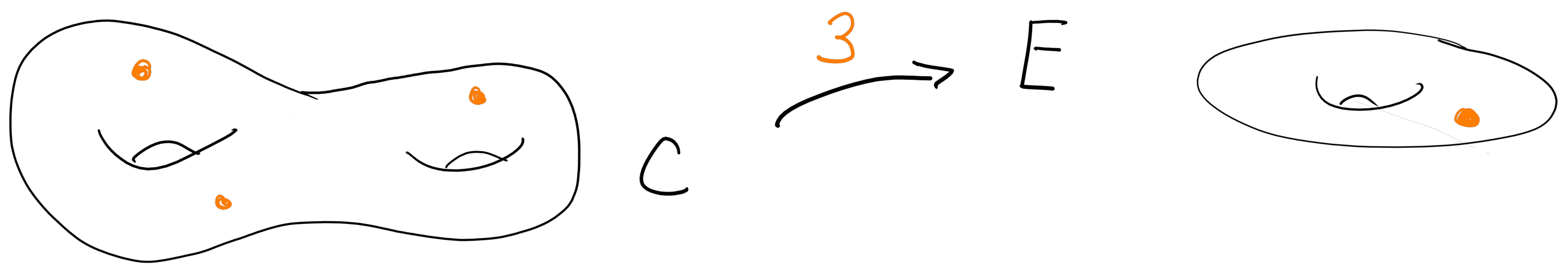
Theorem (Mazur, Kenku): If $\phi: E \to E'$ is a (cyclic) isogeny between elliptic curves defined over $\mathbb{Q}$, then the degree of ϕ is ≤ 19, or $\in \{21,25,27,37,43,67,163\}$.

Upgrade to genus 2

<u>Question:</u> For which integers $N \geq 2$ do there there exist some genus 2 curve $C/\mathbb{Q}$ an elliptic curve $E/\mathbb{Q}$ and a morphism

$$\psi\colon C \to E$$

over $\mathbb{Q}$ and "minimal" degree N?



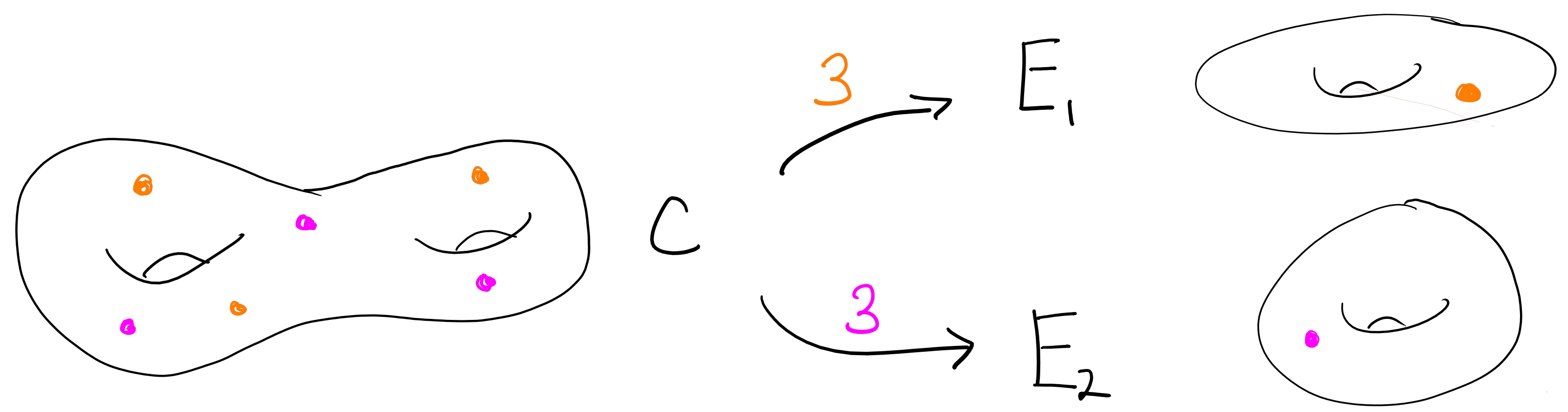
ψ does not factor through an isogeny

For $N \leq 5$ this was thought about a lot in the context of elliptic integrals by Abel—Legendre—Jacobi + (more 19th century authors)

<u>Example:</u> Take $E: y^2 = f(x)$ and $C: y^2 = f(x^2)$ so that we have $C \to E$ induced by $x \mapsto x^2$ with $N = 2$.

In terms of Galois representations

The morphism $\psi\colon C \to E$ induces an isogeny $E \times E' \to \mathrm{Jac}(C)$ over $\mathbb{Q}$ and a different minimal degree N morphism $\psi'\colon C \to E'$.



The existence of $E \times E' \to \mathrm{Jac}(C)$ implies there exists an isomorphism

$$E[N] \cong_{\mathbb{Q}} E'[N]$$

Ensures that the image of $E \times E' \to \mathrm{Jac}(C)$ is a Jacobian

of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$-modules.

The converse also holds so long as E and E' are not isogenous (and the isomorphism $E[N] \cong_{\mathbb{Q}} E'[N]$ is antisymplectic wrt the Weil pairing).

Frey—Mazur conjecture

Conjecture (Frey—Mazur): There exists an integer $N_0 > 0$ such that for all $N \geq N_0$ any pair of elliptic curves $E/\mathbb{Q}$ and $E'/\mathbb{Q}$ with $E[N] \cong_{\mathbb{Q}} E'[N]$ are ~~geometrically~~ isogenous.

Conjecture (Fisher): If one restricts to N is prime, one can take $N_0 = 19$ in the Frey—Mazur conjecture.

Frey—Mazur conjecture

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Theorem (Frey): If the Frey—Mazur conjecture then the a*symptotic Fermat conjecture* holds.

Example / Theorem (F.): There exists a genus 2 curve $C/\mathbb{Q}$ with minimal degree 15 morphism over $\mathbb{Q}$ to

$$E : y^2 + xy + y = x^3 - x^2 - 5978298027424617040871x - 17791581668504438650617886792043 8$$

$$E' : y^2 = x^3 - 21356074373319898419435407107828 11x + 379157831232980070853171470667452834771275433708 06$$

Which are not (geometrically) isogenous and have conductor

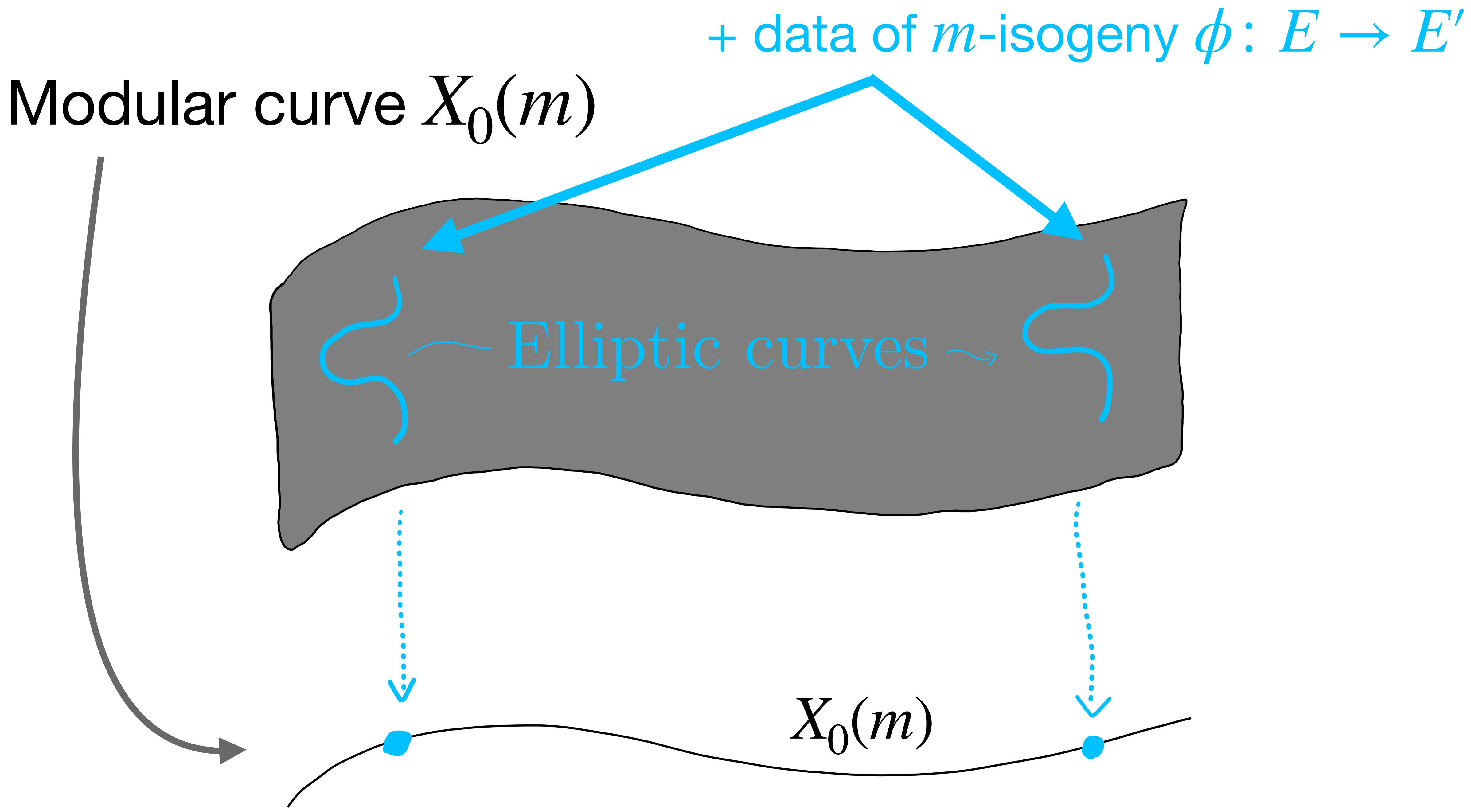
$$3^3 \cdot 7 \cdot 11 \cdot 13 \cdot 23 \cdot 61 \cdot 199^2 \cdot 2341^2 \approx 10^{19}.$$

Also:

- ∞ many for $N \leq 11$ (Kumar + …)
- ∞ many for $N = 12$ (F.)
- ∞ many for $N = 13$ (Fisher)
- ~ 10 for $N = 14$ (F.)
- One for $N = 15$ (F.)
- One for $N = 17$ (Fisher)

The relevant moduli spaces

Isogeny $\phi\colon E \to E'$



Mazur—Kenku "just" find the rational points on $X_0(m)$ for every $m \geq 2$.

Coverings $\psi\colon C \to E$

Hilbert modular surface $Y\_(N^2)$ such that

$\{k\text{-points on } Y\_(N^2)\} \leftrightarrow \{\psi\colon C \to E \text{ over } k\}$

We "just" need to find the rational points on $Y\_(N^2)$ for every $N \geq 2$.

Question: Too hard over $\mathbb{Q}$, make it easier. For which integers N does there exist a family of $\Psi: \mathscr{C} \to \mathscr{E}$ over $\mathbb{C}(s,t)$ (or $\mathbb{Q}(s,t)$)?

Theorem (Hermann, Kani—Schanz): The Hilbert modular surface $Y\_(N^2)$ is (birational to a) surface which is

- Rational if $N \leq 5$
- Elliptic K3 if $N = 6{,}7$
- Properly elliptic if $N = 8{,}9{,}10$
- General type if $N \geq 11$.

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Thm: $\kappa = \min(2, p_g - 1)$!?

Theorem (Bakker—Tismerman): There exists p_0 so that any $E[p] \cong_{\mathbb{C}(t)} E'[p]$ (symplectic isom.) over $\mathbb{C}(t)$ are isogenous when $p \geq p_0$.

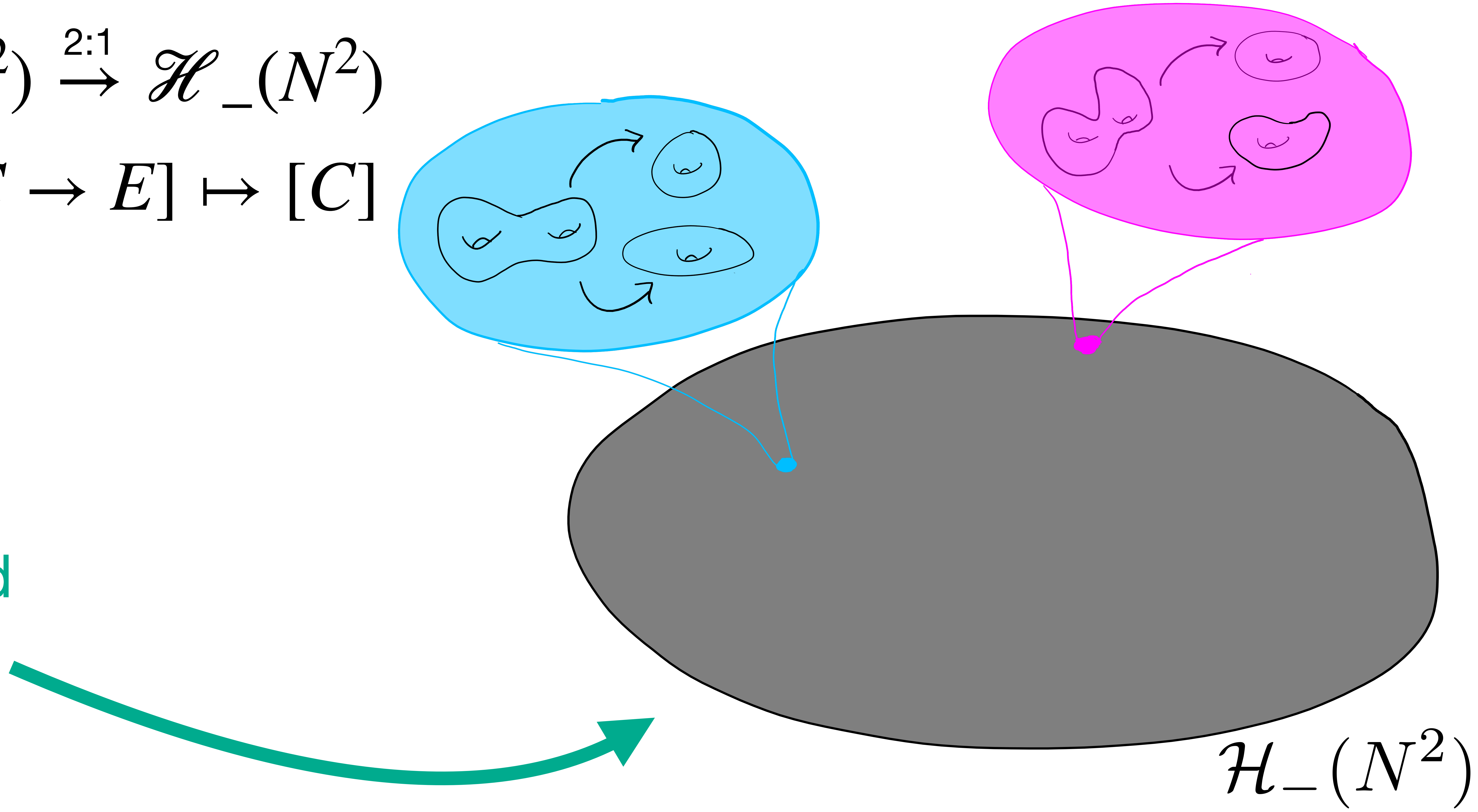
Humbert surfaces

<u>But:</u> We had an example with $N = 15{,}17$? Should this be unexpected since $Y\_(N^2)$ is general type?

$$Y\_(N^2) \xrightarrow{2:1} \mathscr{H}\_(N^2)$$

$$[\psi\colon C \to E] \mapsto [C]$$

Remembers only that C *admits* a map to an elliptic curve (equivalently, unordered pairs $C \to E$ and $C \to E'$)



Humbert surfaces

But: We had an example with $N = 15,17$? Should this be unexpected since $Y_-(N^2)$ is general type?

$$Y_-(N^2) \overset{2:1}{\to} \mathscr{H}_-(N^2)$$

$$[\psi : C \to E] \mapsto [C]$$

Lots of points $N = 15,17$

Probably not many points (Bombieri–Lang conjecture)

Humbert surfaces

Another reason: The surface $\mathscr{H}\_(N^2)$ is very natural from the point of view of moduli. We have a natural way of viewing

$$\mathscr{H}\_(N^2) \subset \mathscr{M}_2$$

As defined it is only birational to its image

The union

$$\bigcup_N \mathscr{H}\_(N^2)$$

are the points in $\mathscr{M}_2$ with Jacobians isogenous to a product of elliptic curves.

As principally polarised abelian varieties

$N = p$

Theorem (Hermann, F.): The Humbert surface of square discriminant $\mathscr{H}\_(N^2)$ is (birational to):

- A rational surface if $N \leq 16$ or if $N = 18, 20,$ or 24,
- An elliptic K3 surface if $N = 17$,
- A properly elliptic surface if $N = 19$ or 21, and
- A surface of general type if $N \geq 22$ and $N \neq 24$.

- Kumar ($N \leq 11$) + …
- Fisher ($N = 13$)
- F. ($N = 12,14,15,18,20,24$)

$\mathscr{C}/\mathbb{Q}(u,v)$

There exists ~~$\mathscr{C}/\mathbb{C}(u,v)$~~ which admits a minimal cover of degree N to an elliptic curve (after base change to a quadratic extension)

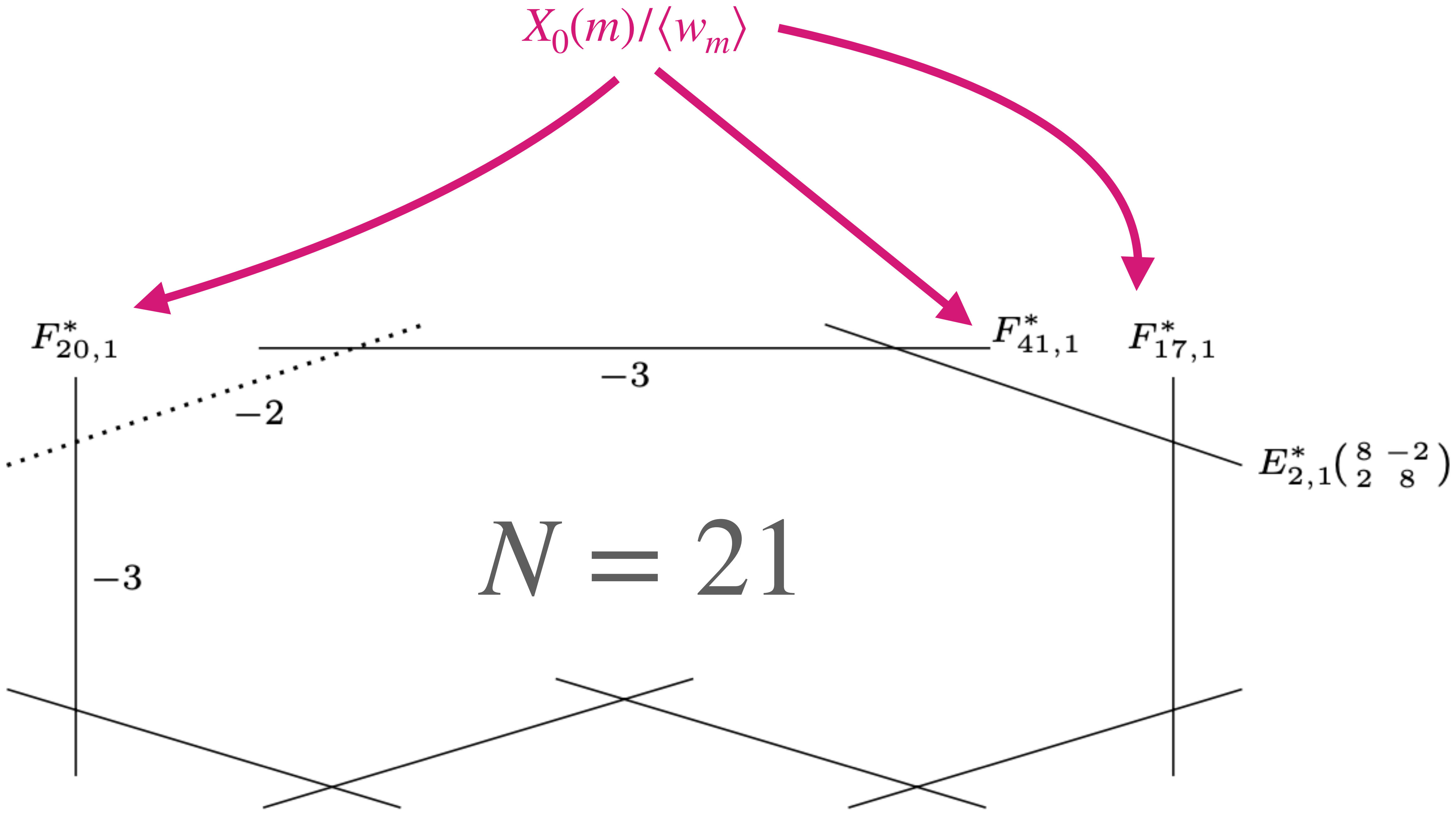
Thm: $\kappa = \min(2, p_g - 1)$!?

How?

and use Riemann—Hurwitz++

Use that $\mathscr{H}\_(N^2)$ is a quotient of $Y\_(N^2)$ by an involution.

1. Understand fixed point locus on $Y\_(N^2)$ (genera, singularities, etc)
   - e.g., $X^+_{\text{ns}}(p^k)$ or $X^+_{\text{s}}(p^k)$ and N odd you get "extended Cartan"
   - N even you get "extended Cartan" and fancier X_H some $H \subset \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$
2. Desingularise quotient of $Y\_(N^2)$
   - blow-down Hecke correspondences (when N even, fancier X_H)
3. Understand (self)-intersections and singularities of Hecke correspondences on quotient to find elliptic fibres etc.



Challenge: Below is $j(\mathscr{E})j(\mathscr{E}') \in \mathbb{Q}(u, v)$ (also have $j(\mathscr{E}) + j(\mathscr{E}')$) for a pair of elliptic curves for which there exists a genus 2 curve $\mathscr{C}/\mathbb{Q}(u, v)$ with degree $N = 24$ maps to elliptic curves (over a quadratic extension of $\mathbb{Q}(u, v)$). **Find $\mathscr{C}$.**